8/05/2014 Lec 24

Local extrema

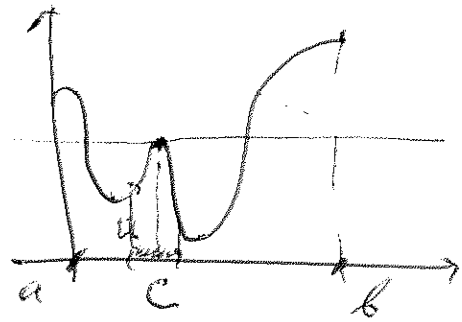
A function $f: E \to \mathbb{R}$ has a <u>local maximum</u> at $c \in E$ if there exists an open interval $U \subset \mathbb{R}$ s.t. $c \in U$ and

$$f(c) \geq f(x)$$

for all $x \in U \cap E$. (Strict maximum if $>$).

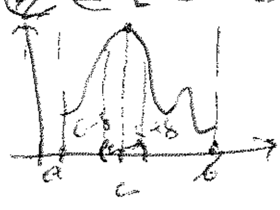
Local minimum - analogously.

c is called a local extremum if it is a local max <u>or</u> a local min.



<u>Theorem.</u> Suppose $f: [a,b] \to \mathbb{R}$ is a function, $c \in (a,b)$. If f has a loc. extremum at c and $f'(c)$ exists, then $f'(c) = 0$.

<u>Proof.</u> Without loss of generality assume c is a local max (otherwise consider $-f$). There exists $\delta > 0$ s.t. $[c-\delta, c+\delta] \subset [a,b]$ and $f(x) \leq f(c)$ for all $x \in [c-\delta, c+\delta]$.



Consider the function

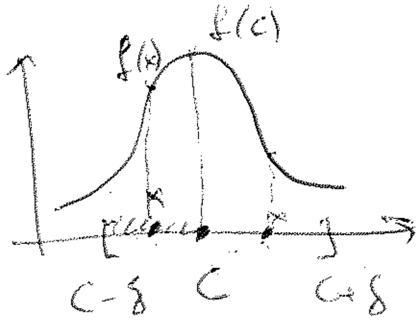
$$q(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c}, & x \neq c, \\ f'(c), & x = c. \end{cases}$$

This function is continuous at c.

When $x \in [c-\delta, c]$, $q(x) \geq 0$.

Also, when $x \in [c, c+\delta]$,

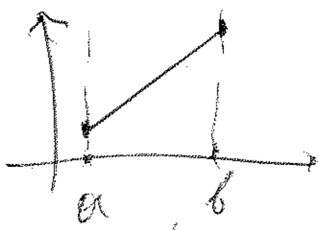
$q(x) \leq 0$.



By continuity, this implies $q(c) = 0$. But $q(c) = f'(c)$.

Thus, $f'(c) = 0$. $\square$

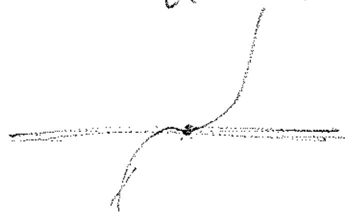
Remarks. 1) If $c = a$ or $c = b$, then $f'(c)$ is not necessarily 0.



2) If $f(x) = x^3$, then

$f'(0) = 3x^2\big|_{x=0} = 0$.

But 0 is not a local extr.

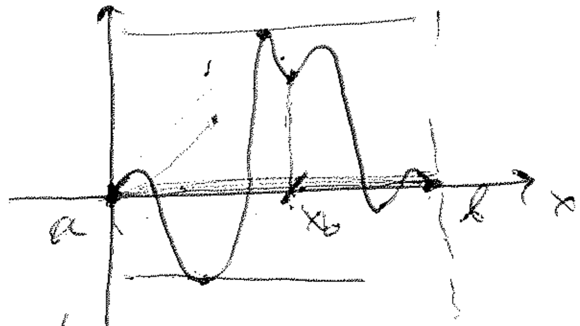


3) $f(x) = |x|$, $x = 0$ is a loc. min.

But $f'(0)$ does not exist (certainly $\neq 0$).

Theorem (Rolle's theorem).

Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ s.t. $f'(c) = 0$.



Proof. If $f(x) = f(a)$ for all $x \in [a, b]$, then $f' = 0$ on (a, b), and we're done.

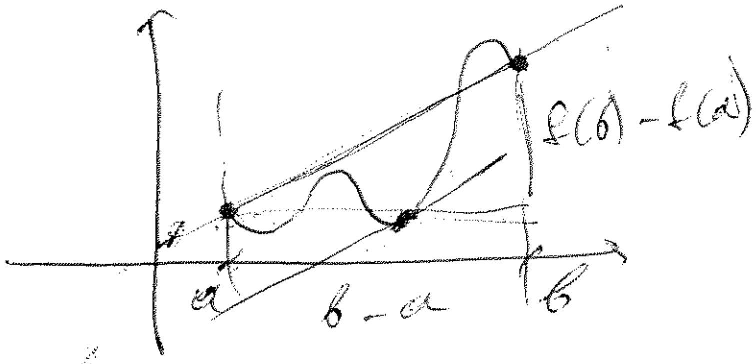
If not, assume there exists $x_0 \in (a, b)$ s.t. $f(x_0) > f(a) = f(b)$ (if no such x_0 exists, simply consider $-f$).

Now, f achieves its max on $[a, \cancel{x_0}\, b]$. This max is not at a or b. Therefore, it is ~~at~~ in $(a, \cancel{x_0}\, b]$, and f is diff on $(a, \cancel{x_0}\, b]$. Therefore, $f' = 0$ at this max by the previous theorem. $\square$

~~(THERE IS A PROBLEM! FIX TOMORROW)~~

Theorem (mean value theorem - MVT):
Suppose $f:[a,b]\to\mathbb{R}$ is cont. on $[a,b]$, diff. on (a,b). Then there exists $c\in(a,b)$ s.t.

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$



Proof. Take the function

$$\phi(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a).$$

Apply Rolle. $\square$

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Applications of the mean value theorem

1) Assume $f:[a,b]\to\mathbb{R}$ is cont. on $[a,b]$ and diff. on (a,b). If $f'(x)=0$ for all $x\in(a,b)$, then $f=\text{const.}$ on $[a,b]$.

Take $x \in (a,b]$. Apply MVT on $[a,x]$.

Then

$$f(x)-f(a) = f'(\xi)(x-a) = 0, \quad \boxed{\xi}$$

for some $\xi \in (a,x)$.

Thus $f(x) = f(a)$ for all $x \in (a,b]$.

2) Assume $|f'(x)| \le M$ for all $x \in (a,b)$.

Then $|f(x)-f(y)| \le M|x-y|$

for all $x, y \in (a,b)$.

<u>Proof.</u> Without loss of generality, assume $y > x$. Apply MVT on $[x,y]$. Obtain

$$f(y)-f(x) = f'(\xi)(y-x) \text{ for some } \xi \in (x,y).$$

This implies

$$|f(y)-f(x)| = |f'(\xi)|\,|y-x| \le M|y-x|.$$

□

<u>Theorem.</u> Assume f is cont. on $[a,b]$, diff on (a,b).

1) If $f' \ge 0$ on (a,b), then f is nondecreasing.

2) If $f' \leq 0$, then f is nonincreasing.

3) If $f' > 0$, then f is strictly incr.

4) If $f' < 0$, then f is strictly decr.

<u>Proof.</u> 1) Take $x, y \in [a, b]$ s.t. $x < y$. Need to show $f(x) \leq f(y)$. Apply MVT on $[x, y]$:

$$f(y) - f(x) = \underbrace{f'(\xi)}_{\geq 0} \underbrace{(y - x)}_{> 0} \geq 0, \text{ so}$$

$$f(y) \geq f(x).$$

$\square$

2), 3), 4) - analogously.

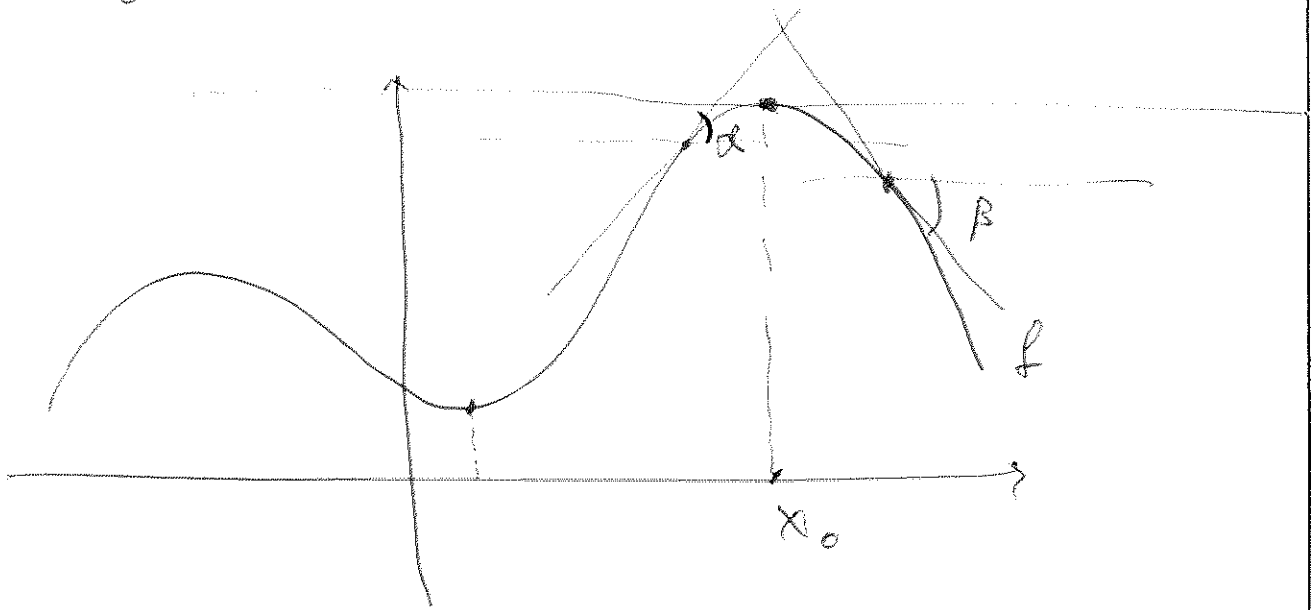
<u>Note.</u> The converse to 3), 4) would be false in general. Take $f(x) = x^3$ (or $f(x) = -x^3$). Clearly, f is strictly incr. on $\mathbb{R}$. However, $f'(0) = 0$, which is not positive.

<u>Theorem.</u> Take $f : [a, b] \to \mathbb{R}$ contin. on $[a, b]$. Assume f is differentiable <u>twice</u> on (a, b). Let $x_0 \in (a, b)$.

Then

1) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a local min.

2) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a loc. max.



<u>Proof.</u> 1) By definition,

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - \overbrace{f'(x_0)}^{=0}}{x - x_0} = \lim_{x \to x_0} \frac{f'(x)}{x - x_0}$$

and $\displaystyle\lim_{x \to x_0} \frac{f'(x)}{x - x_0} > 0$.

Because f' is differentiable at x_0, it is continuous. Therefore,

$$\frac{f'(x)}{x - x_0} > 0 \text{ in } (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$$

for some $\delta > 0$. Hence $f'(x)$ is negative on $(x_0 - \delta, x_0)$.

Similarly, $f'(x) > 0$ on $(x_0, x_0+\delta)$.

Therefore, f ~~is~~ decreasing on $(\cancel{x_0}\, x_0-\delta, x_0)$ and incr. on $(x_0, x_0+\delta)$. This means f attains its min at x_0 on $(x_0-\delta, x_0+\delta)$ (by the previous theorem). Thus, f has a local min at x_0.

$\square$

2) analogously.

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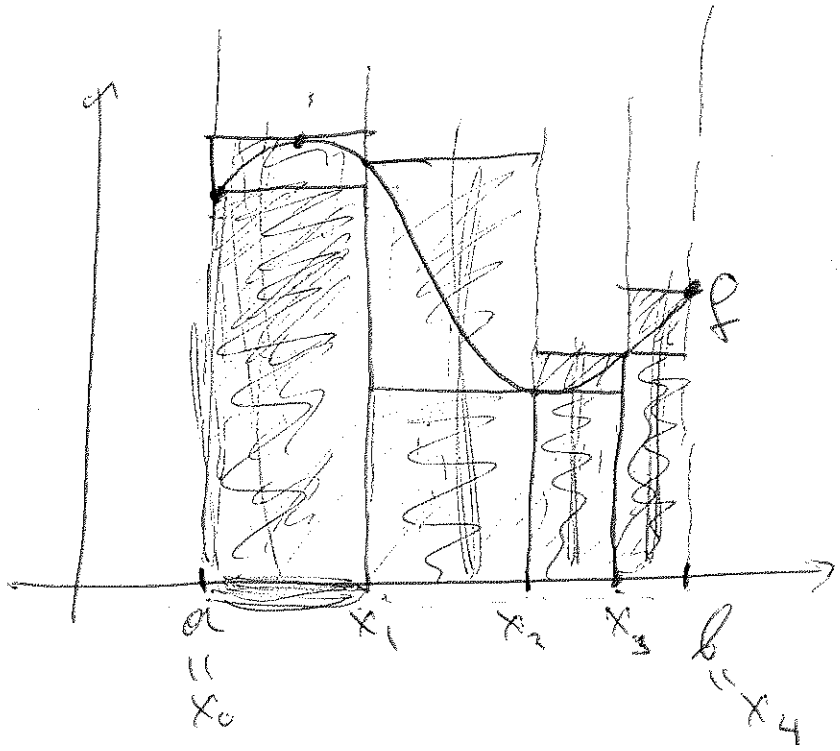
(Riemann) integration.

Assume $f : [a,b] \to \mathbb{R}$ is bounded

(not necessarily continuous).

Def. A partition P of $[a,b]$ is a set of points $\{x_0, x_1, \dots, x_{n-1}, x_n\}$ s.t.

$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$.



The lower sum of f w.r.t. P is

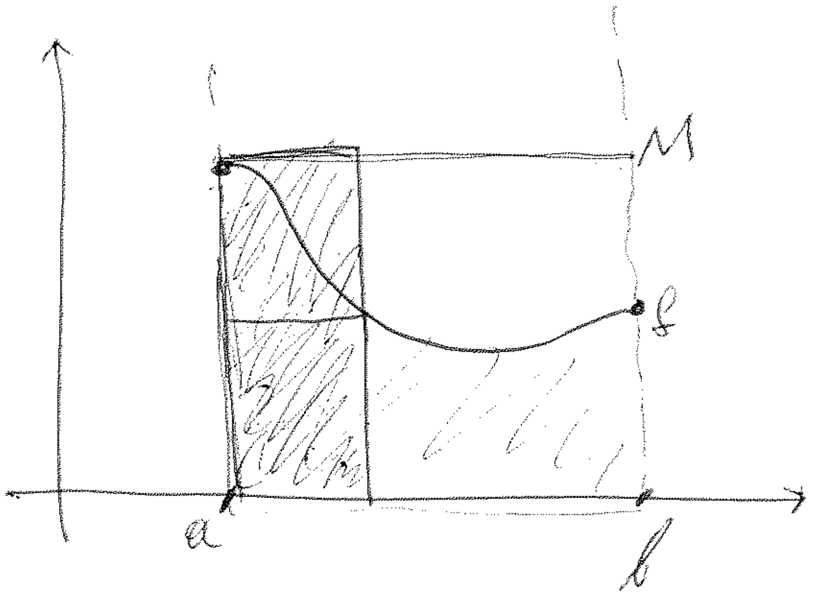
$L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$, where $m_i := \inf_{x \in [x_i, x_{i-1}]} f(x)$

The upper sum is

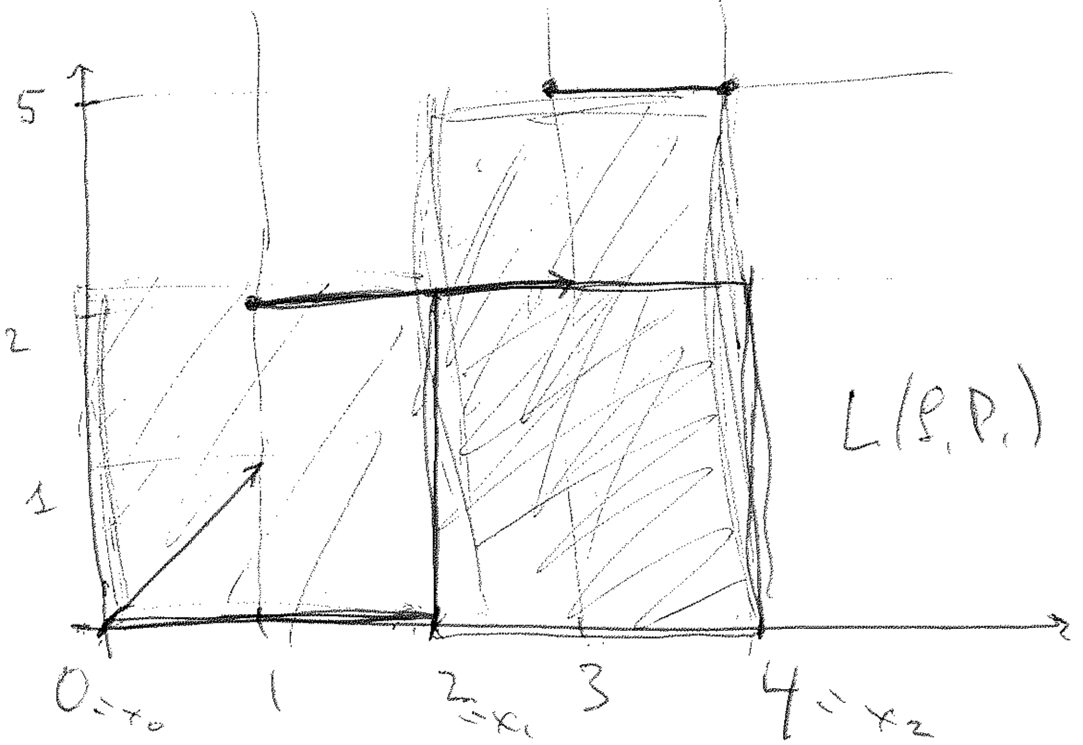
$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$, $M_i := \sup_{x \in [x_i, x_{i-1}]} f(x)$

Note. if $|f| \le M$, then

$$-M(b-a) \le L(f,P) \le U(f,P) \le M(b-a).$$



Ex. $f(x) = \begin{cases} x, & x \in [0,1), \\ 2, & x \in [1,3), \\ 5, & x \in [3,4], \end{cases}$ on $[0,4]$.



Take $P_1 = \{0, 2, 4\}$.

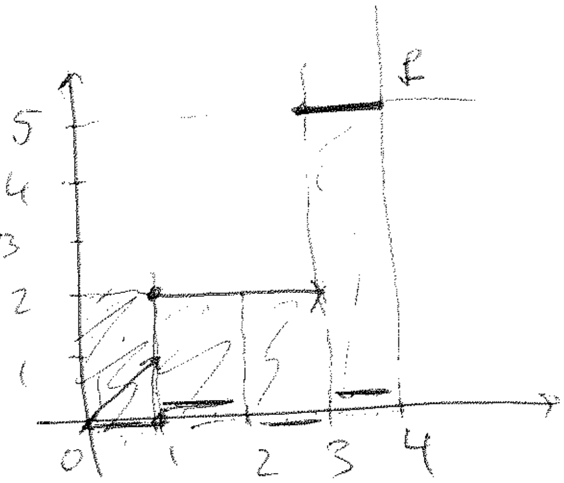
Find $L(f,P_1)$: $L(f,P_1) = \sum_{i=1}^{2} \inf_{[x_{i-1},x_i]} f \, (x_i - x_{i-1})$

$= 0 \cdot (2-0) + 2 \cdot (4-2) = 4$.

Now,

$$\underline{U(f, P_1) = 2 \cdot (2-0) + 5(4-2) = 14.}$$

Now, take $P_2 = \{0, 1, 2, 3, 4\}$.



$L(f, P_2)$

$= 0 \cdot (1-0) + 2(\cancel{1}2-1)$

$+ 2(3-2)$

$+ 5(4-3) = 9$.

$U(f, P_2) = 2 \cdot 1 + 2 \cdot 1 + 5 \cdot 1 + 5 \cdot 1 = 14$.

<u>Def.</u> The partition P' is a <u>refinement</u> of P if $P' \supset P$.

<u>Lemma.</u> If $P' \supset P$, then

$L(f, P) \leq L(f, P')$ and $U(f, P) \geq U(f, P')$.

<u>Proof.</u> Let P equal $\{x_0, x_1, \ldots, x_n\}$ and

$P' = \{x_0, x_1, x_2, \ldots, x_{j-1}, y, x_j, \ldots, x_n\}$.

Then

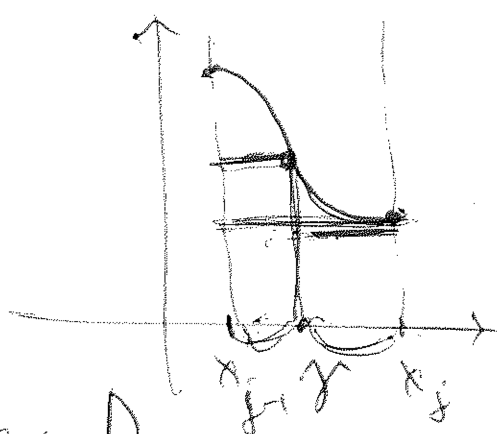
$$\underline{L(f,P)} = \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f(x)\cdot(x_i - x_{i-1})$$

$$= \sum_{\substack{i=1\\ i\neq j}}^{n} \inf_{[x_{i-1},x_i]} f(x)\,(x_i - x_{i-1}) + \inf_{[x_{j-1},x_j]} f(x)\,(x_j - x_{j-1})$$

$$\leq \sum_{\substack{i=1\\ i\neq j}}^{n} \inf_{[x_{i-1},x_i]} f(x)\,(x_i - x_{i-1}) + \inf_{[x_{j-1},y]} f(x)\,(y - x_{j-1})$$

$$+ \inf_{[y,x_j]} f(x)\,(x_j - y)$$

$$= \underline{L(f,P')}.$$



Now, if P' differs from P by m points, just repeat this procedure m times. The ineq. for $U(f,P)$ is analogous. $\square$

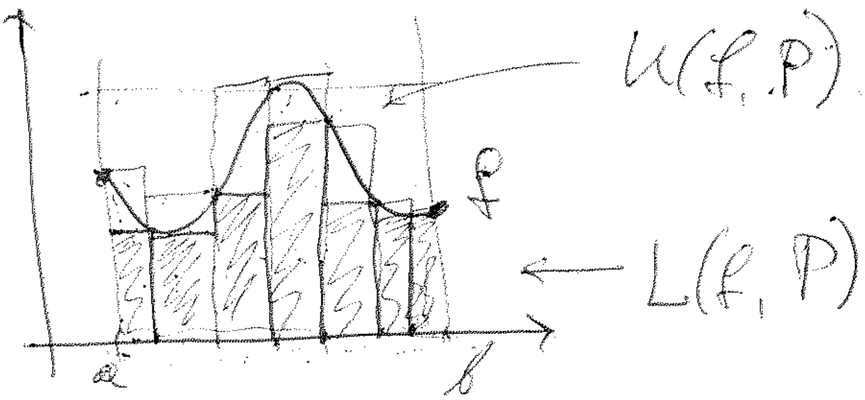
Def. The number

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$$

is called the <u>lower integral</u> of f over $[a,b]$

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$$

is the <u>upper integral</u>.



<u>Lemma</u>. If P_1 and P_2 are partitions of $[a,b]$, then $L(f,P_1) \le U(f,P_2)$.

<u>Proof</u>. Set $P = P_1 \cup P_2$. Then

$$L(f,P_1) \le L(f,P) \le U(f,P) \le U(f,P_2).$$

$\square$

This lemma implies $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$.

Def. If $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$, then f is called integrable and
$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)\,dx.$$

Theorem. $f : [a,b] \to \mathbb{R}$ is integrable iff for every $\varepsilon > 0$ there is P s.t.
$$U(f,P) - L(f,P) < \varepsilon.$$

Proof. $(\Rightarrow)$ exercise.

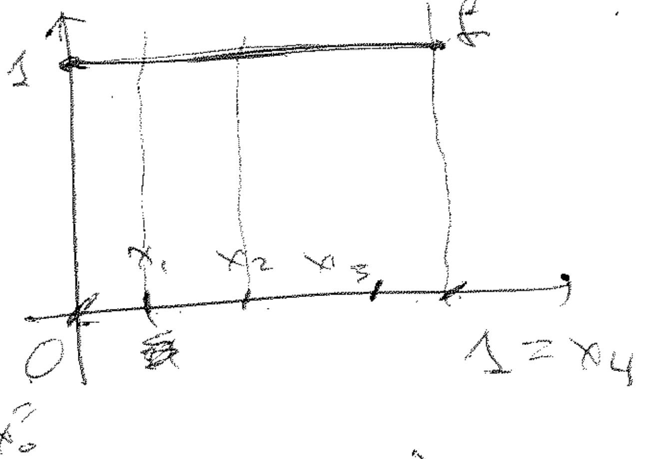
$(\Leftarrow)$ Fix $\varepsilon > 0$ and the corresponding partition P. Then
$$\overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx \le U(f,P) - L(f,P) < \varepsilon.$$

Thus, $0 \le \overline{\int_a^b} - \underline{\int_a^b} < \varepsilon$ for <u>all</u> $\varepsilon > 0$.

Thus, $\overline{\int} = \underline{\int}$ and f is integrable. $\square$

Examples. ① $f(x) = 1$ for all $x \in [0,1]$

Take any partition P of $[0,1]$.
" $\{x_0, x_1, \ldots, x_n\}$.



Then

$$L(f,P) = \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f(x)\,(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} 1\,(x_i - x_{i-1})$$

$$= (\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + (x_3 - \cancel{x_2}) + \cdots + (\cancel{x_{n-1}} - x_{n-2}) + (x_n - \cancel{x_{n-1}})$$

$$= x_n - x_0 = 1.$$

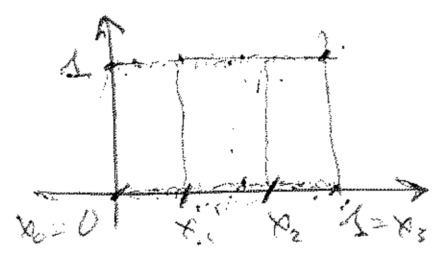
This means $\underline{\int_a^b} f(x)\,dx = 1$.

Similarly, $\overline{\int_a^b} f(x)\,dx = 1$.

Thus, f is integrable and $\int_0^1 f(x)\,dx = 1$.

② Define $f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Take and partition $P = \{x_0, \ldots, x_n\}$ of $[0,1]$.



$$L(f,P) = \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) = 0.$$

Now,

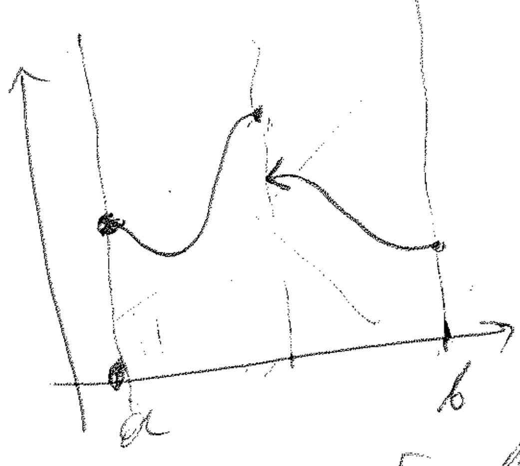
$$U(f,P) = \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f(x)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} 1(x_i - x_{i-1}) = 1.$$

Thus, $\underline{\int_0^1} f(x)\,dx = 0$, $\overline{\int_0^1} f(x)\,dx = 1$.

<u>Theorem.</u> 1) If $f:[a,b] \to \mathbb{R}$ is bounded and continuous at all but finitely many points, then f is integrable on $[a,b]$

2) If it is increasing or decreasing, then f is integrable.



<u>Proof.</u> Part 1).

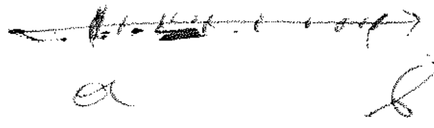
<u>Case A.</u> f is continuous on $[a,b]$.

Then f is uniformly continuous on $[a,b]$.

Fix $\varepsilon > 0$. There exists $\delta > 0$ s.t. if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon/(b-a)$.

Take a partition P of $[a,b]$ s.t. $P = \{x_0, x_1, \dots, x_n\}$ and $|x_i - x_{i-1}| < \delta$ for all $i = 1, \dots, n$.

Then



$$U(f,P) - L(f,P)$$

$$= \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f \,(x_i - x_{i-1}) - \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f \,(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \Big(\overset{f(x)}{\sup_{[x_{i-1},x_i]} f} - \overset{f(y)}{\inf_{[x_{i-1},x_i]} f} \Big) (x_i - x_{i-1})$$

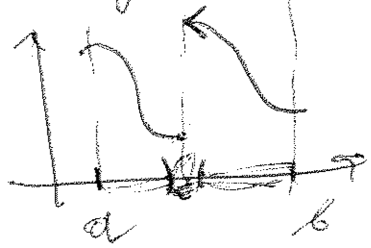
$$\leq \sum_{i=1}^{n} \frac{\varepsilon}{b-a} \cdot (x_i - x_{i-1})$$

$$= \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\varepsilon}{\cancel{b-a}} \cdot \cancel{(b-a)}$$

$$= \varepsilon.$$

This means f is integrable.

<u>Case B</u>. f has exactly one discontinuity.

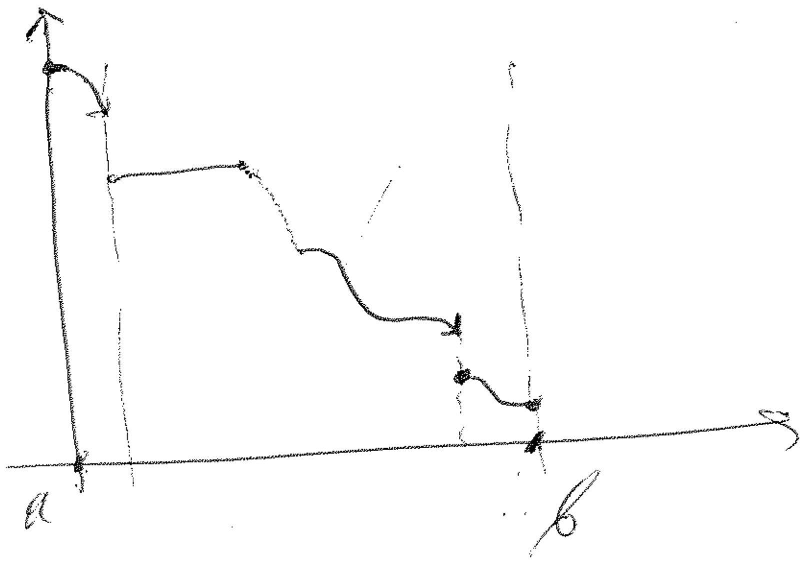


Exercise.

<u>Case C</u>. Finitely many discont. Exercise.

Proof of part 2. 16/05/14 Lec 27.

<u>Reminder.</u> If $f: [a,b] \to \mathbb{R}$ is increasing or decreasing (not necessarily strictly), the f is integrable.



<u>Proof.</u> Assume f is decreasing (otherwise consider $-f$).

Fix $\varepsilon > 0$. We want to find a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

Let P be the partition of $[a,b]$ into n equal parts. Thus, $x_k = a + \frac{b-a}{n} \cdot k$, $k = 0, 1, \ldots, n$